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LETTER TO THE EDITOR

A new systematic approach to the decay on an unstable state

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Abstract. To describe a decay on an unstable state, a new systematic procedure is proposed for solving the Fokker–Planck equation with a small noise strength. The approximation assumes the correct small-time evolution and the Gaussian fluctuations around the most probable path for larger times. The exact stationary distribution is reached. The method may be easily adapted for calculating the multi-time correlation functions.

In recent years the temporal behaviour of a system initially prepared in an unstable state has been investigated very intensively. The approaches include numerical [1–3], strictly analytical [4–7], analytical with some empirical parameters [8] and stochastic [9] methods. The complexity of the problem lies in the fact that it is the fluctuations that initiate the evolution so they must be considered simultaneously with the deterministic forces.

The state of a system may be described by a probability density function $W(\tilde{x}, t)$ the evolution of which is given by the Fokker–Planck (FP) equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial \tilde{x}} U'(\tilde{x}) W + q \frac{\partial^2}{\partial \tilde{x}^2} W \tag{1}$$

where q is a diffusion constant and $-\infty < \tilde{x} < +\infty$. In the simplest case the potential $U(\tilde{x})$ has the (symmetrical) form

$$U(\tilde{x}) = \tfrac{1}{4}\tilde{x}^4 - \tfrac{1}{2}\tilde{x}^2. \tag{2}$$

Starting from its unstable state $\tilde{x} = 0$, the system tends to one of its potential minima $\tilde{x} = \pm 1$. Because of the noise there is a finite probability of finding the system in each potential well.

There are some physical situations (e.g. in quantum optics) in which the variable describing the system is non-negative. In such a case the simplest model corresponding to (1) and (2) is given by a FP equation

$$\frac{\partial W(x, t)}{\partial t} = L(x) W(x, t) \tag{3}$$

with a FP operator $L(x)$

$$L(x) = 2\frac{\partial}{\partial x} x(x-a) + 4q \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \tag{4}$$

where $x = \tilde{x}^2$. Now the system evolves from its unstable state $x = 0$ to the single stable state $x = 1$.

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Almost all [4-6] of the analytical methods mentioned above deal with equations (1) and (2). Among them Haake [6] neglects the stationary fluctuations and Suzuki [5] outlines a method rather than giving a precise recipe (there is an arbitrarily chosen function in his final expression). Mizerski [7] formulates a method which seems to work quite well (compared with others) in the critical region but yields an incorrect small-time description. In this letter some analytic formulae providing the correct small-time and exact stationary description of (3) are proposed for the small-noise ($q \ll 1$) approximation.

A formal solution of (3) is

$$W(x, t) = \exp(L(x)t)\, W(x, 0) \tag{5}$$

with the initial condition for the probability density

$$W(x, 0) = \lim_{x' \to 0^+} \delta(x - x'). \tag{6}$$

The evolution of the kth moment is thus given by

$$\langle x^k \rangle = \int_0^\infty \mathrm{d}x\, x^k \exp(L(x)t)\, W(x, 0). \tag{7}$$

As is known (e.g. [3]) the linearised FP operator describes well the very beginning of evolution—it completely determines the first two terms of a time power expansion of the kth moment

$$\langle x^k \rangle = k!(4qt)^k + kk!(4q)^k t^{k+1} + \mathrm{O}(t^{k+2}). \tag{8}$$

In such a case the evolution operator of (5) may be approximated as

$$\begin{aligned} \exp[L(x)t] &\simeq \exp\left[\left(-2\frac{\partial}{\partial x}x + 4q\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)t\right] \\ &= \exp\left(-2\frac{\partial}{\partial x}xt\right)\exp(L_{\mathrm{in}}(x, t)) \end{aligned} \tag{9}$$

with

$$L_{\mathrm{in}}(x, t) = 2q(1 - \mathrm{e}^{-2t})\frac{\partial}{\partial x}x\frac{\partial}{\partial x} \equiv q\rho(t)\frac{\partial}{\partial x}x\frac{\partial}{\partial x}. \tag{10}$$

The factorisation made in (9) separates the linearised drift operator and the diffusion constant q. Thus for the initial regime and for the intermediate time regime, where (far from the critical region) the evolution may be described strictly deterministically, the evolution operator will be approximated by

$$\exp[L(x)t] \simeq \exp(L_0(x)t)\exp(L_{\mathrm{in}}(x, t)) \tag{11}$$

where $L_0(x)$ is the drift part of the FP operator. This expression is just the quasideterministic theory of de Pasquale *et al* [9] in the FP equation language. The inconvenience of that treatment is that the fluctuations are taken into account only at the beginning of the evolution—they are completely neglected in the description of the stationary state.

In the following we somewhat modify this theory to obtain the exact stationary distribution function. As it results from the above consideration, the two operators standing on the RHS of (11) describe well the initiating role of the fluctuations and the intermediate time regime of evolution. Therefore we can extract them to the right from the total evolution operator in order to act with them directly on the initial distribution (6)

$$\exp(L(x)t) = T(x, t)\exp(L_0(x)t)\exp(L_{\text{in}}(x, t)) \tag{12}$$

where

$$T(x, t) = \underset{\sim}{T}\exp\left(\int_0^t \mathrm{d}s\tilde{L}(x, s)\right) \tag{13}$$

where $\underset{\sim}{T}$ denotes the chronological operator ordering from the right to the left as time increases. The operator $\tilde{L}(x, s)$ is

$$\begin{aligned}\tilde{L}(x, s) = {} & \exp(L_0(x)s)[\exp(L_{\text{in}}(x, s))(L(x) \\ & - \dot{L}_{\text{in}}(x, s))\exp(-L_{\text{in}}(x, s)) - L_0(x)]\exp(-L_0(x)s) \\ = {} & 2q\rho(s)\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s)\left(3\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s) - 1\right) \\ & + 2q^2\rho^2(s)\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s)\frac{\partial}{\partial x}\psi(x, -s) \\ & \times\left(3\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s) - 2\right) \\ & + 2q^3\rho^3(s)\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s)\frac{\partial}{\partial x}\psi(x, -s) \\ & \times\frac{\partial}{\partial x}\psi(x, -s)\phi(x, -s)\frac{\partial}{\partial x}\psi(x, -s).\end{aligned} \tag{14}$$

Here the dot denotes the time derivative. The functions $\phi(x, t)$ and $\psi(x, t)$ are given by the differential equations:

$$\dot{\phi} = -2\phi(\phi - 1) \qquad \phi(x, 0) = x \tag{15}$$

$$\dot{\psi} = 2(2\phi - 1)\psi \qquad \psi(x, 0) = 1. \tag{16}$$

Expression (15) is the deterministic equation with the solution

$$\phi(x, t) = x/[x + (1 - x)\,\mathrm{e}^{-2t}] \tag{17}$$

and the function $\psi(x, t)$ is given as the inverse of $\partial\phi/\partial x$. The functions ϕ and ψ occuring in (14) are taken for negative time $-s$ for which the solution of (15) may not exist. However, one must treat them rather formally because in the final expression the operator $\exp(L_0(x)t)$ acts on $T(x, t)$ (see (12)) changing the x variable by $\phi(x, t)$ while, as follows from (13), $s \leqslant t$. Thus in (14) we deal with functions ϕ and ψ for the time moment $-s$ but with an initial condition $\phi(x, t)$ so one is always on the positive part of the deterministic trajectory.

From the comparison of (12) and (11) it follows that the operator $T(x, t)$ describes the fluctuation of the final and partially of the intermediate time regimes. Inserting the unity operator $\int_0^\infty \mathrm{d}y\,\delta(x - y)$ into the RHS of (12) and introducing a new variable

$$z = \varepsilon^{-1}(x - \phi(y, t)) \qquad \varepsilon = q^{1/2} \tag{18}$$

we have

$$\begin{aligned}\exp(L(x)t) &= T(x, t)\int_0^\infty \mathrm{d}y\,\delta(x - y)\exp(L_0(y)t)\exp(L_{\text{in}}(y, t)) \\ &= \int_0^\infty \mathrm{d}y\,T(y + \varepsilon z, t)\delta(\varepsilon z)\exp(L_0(y)t)\exp(L_{\text{in}}(y, t)).\end{aligned} \tag{19}$$

Since the operator $\exp(L_0(y)t)$, acting to the left on the variable y, gives a deterministic trajectory $\phi(y, t)$, the new variable z means a rescaled deviation from it. Thus if we are far enough from the critical region the operator $\hat{L}(z, y, s) = \tilde{L}(y+\varepsilon z, s)$ may be expanded in powers of ε

$$\hat{L}(z, y, s) = 6\rho(s)\psi^2(y, -s)\phi^2(y, -s)\partial^2/\partial z^2 + \mathrm{O}(\varepsilon). \tag{20}$$

The lowest-order approximation of the operator $\hat{T}(z, y, t) = T(y+\varepsilon z, t)$ is thus given as

$$\hat{T}(z, y, t) = \exp\left(\tfrac{1}{2}\sigma(y, t)\frac{\partial^2}{\partial z^2}\right) \tag{21}$$

where

$$\begin{aligned} \sigma(y, t) &= \int_0^t \mathrm{d}s\; 12\rho(s)\psi^2(y, -s)\phi^2(y, -s) \\ &= 24y^2[(1-y)^2 t + \tfrac{1}{2}(1-y)(3y-1)(1-\mathrm{e}^{-2t}) \\ &\quad + \tfrac{1}{4}y(3y-2)(1-\mathrm{e}^{-4t}) - \tfrac{1}{6}y^2(1-\mathrm{e}^{-6t})]. \end{aligned} \tag{22}$$

One may easily notice that $\sigma(y, t)$ is positive for all positive y and t. Summarising all the above, the approximate expression for the formal solution of the exact problem (5) is

$$\begin{aligned} W(x, t) &\simeq \int_0^\infty \mathrm{d}y \exp\left(\tfrac{1}{2}q\sigma(y, t)\frac{\partial^2}{\partial x^2}\right)\delta(x-y)\exp(L_0(y)t)\exp(L_{\mathrm{in}}(y, t))\,W(y, 0) \\ &= :\left\{\exp\left(\tfrac{1}{2}q\frac{\partial^2}{\partial x^2}\sigma(x, t)\right)\right\}: \exp(L_0(x)t)\exp(L_{\mathrm{in}}(x, t))\,W(x, 0) \end{aligned} \tag{23}$$

where : : means the ordering operation (all the differentiation operators $\partial/\partial x$ stand to the left). Thus the evolution operator is factorised into three parts. The role of each of them may be better seen if one considers the expressions for the distribution function and its moments.

The smallness of ε gives the possibility to write the distribution function $W(x, t)$ (23) approximately. The variable z belongs to the interval $[-\phi/\varepsilon, +\infty)$. If the lower boundary is $-\infty$ the operator $\hat{T}(z, y, t)$ acts on $\delta(\varepsilon z)$ giving a Gaussian function with a zero mean and $\sigma(\phi, t)$ as a variance. Since, however, this lower boundary is still finite, we obtain a Gaussian-like function with the normalisation constant

$$N^{-1}(y, t) = \left(\frac{\pi\sigma(y, t)}{2}\right)^{1/2}\left[1 + \operatorname{erf}\left(\frac{y}{\varepsilon(2\sigma(y, t))^{1/2}}\right)\right]. \tag{24}$$

Here erf is the error function [10]. Now the distribution function takes the form

$$W(x, t) = \left\langle N(\phi, t)\exp\left(-\frac{(x-\phi)^2}{2q\sigma(\phi, t)}\right)\right\rangle_{\mathrm{in}}. \tag{25}$$

The bracket $\langle\ \rangle_{\mathrm{in}}$ means averaging over the probability distribution

$$W_{\mathrm{in}}(y, t) = \exp(L_{\mathrm{in}}(y, t))\,W(y, 0) = \frac{1}{q\rho}\exp\left(-\frac{y}{q\rho}\right) \tag{26}$$

reflecting the influence of the fluctuations, which begin the evolution, on the initial distribution function. We can easily observe that this expression gives the correct stationary solution of (3). The above Gaussian approximation does not disturb the previous ε expansion since, as long as we are far from the critical region ($q \ll 1$), it creates the corrections of order of $\exp(-1/q)$.

The first two moments obtained from (25) are

$$\langle x \rangle = \left\langle \phi + q\sigma(\phi, t) N(\phi, t) \exp\left(-\frac{\phi^2}{2q\sigma(\phi, t)}\right)\right\rangle_{\text{in}} \tag{27}$$

$$\langle x^2 \rangle = \left\langle \phi^2 + q\sigma(\phi, t) + q\phi\sigma(\phi, t) N(\phi, t) \exp\left(-\frac{\phi^2}{2q\sigma(\phi, t)}\right)\right\rangle_{\text{in}}. \tag{28}$$

If one uses expression (23) for the calculation of the moments one will obtain formulae like (27) and (28) but without the terms including the normalisation constant $N(\phi, t)$. However, as mentioned above, these terms do not contribute essentially.

Returning to the discussion of the expression (23) we can see that the third operator on its RHS gives an averaging over $W_{\text{in}}(y, t)$, the second one introduces the deterministic trajectory $\phi(y, t)$ for each initial point y, and the first one gives Gaussian fluctuations around this trajectory with the variance $q\sigma(\phi(y, t), t)$. One may check that for small time the evolution is governed really by the operator $\exp(L_{\text{in}}(y, t))$ and the linearised operator $\exp(L_0(y)t)$. Since, as time tends to infinity, the deterministic trajectory $\phi(y, t)$ depends less and less on the initial condition y, the influence of $W_{\text{in}}(y, t)$ on the final expression becomes smaller and smaller. On the other hand, the contribution of the function $\sigma(\phi, t)$ continuously increases, leading to the correct stationary description.

The time dependence of the mean $\langle x \rangle$ and its variance $\langle (\Delta x)^2 \rangle$, given by the formulae (27) and (28), are plotted in figures 1 and 2 for two values of the diffusion constant q. The present results are compared with the numerical ones [1, 3]. As one expects, for small q the agreement between the two methods is quite good and the differences occur only in the intermediate time region.

The inclusion of higher-order terms of the operator $\hat{L}(z, y, s)$ (20) should correct the results. For increasing q the discrepancy arises due to invalidity of the assumption

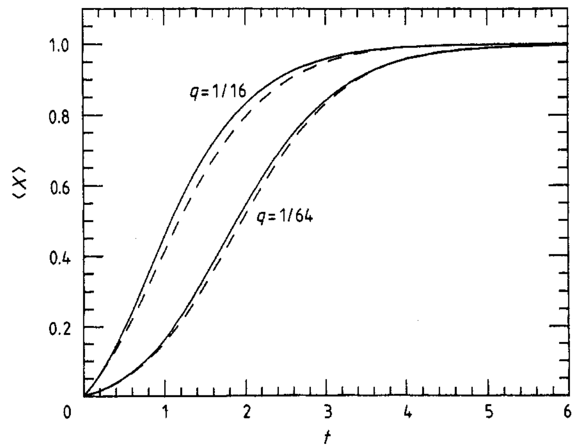

Figure 1. Time dependence of the mean value $\langle x \rangle$ for the diffusion constant $q = \frac{1}{16}$ and $q = \frac{1}{64}$ (these values correspond respectively to the values $a = 4$ and $a = 8$ of the Risken parameter a [1]). The full curve represents the numerical result [1, 3] and the broken curve is the present result of equation (27).

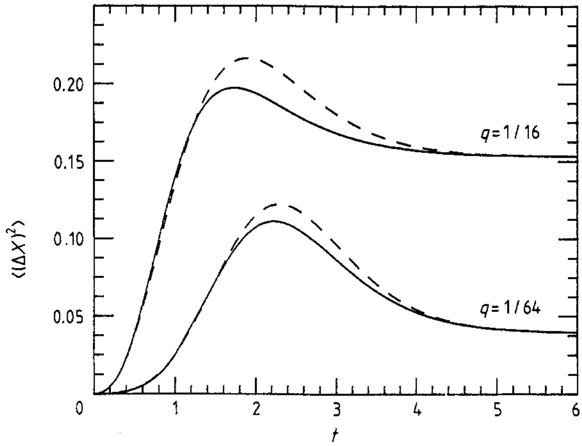

Figure 2. The same as in figure 1 but for the variance $\langle(\Delta x)^2\rangle$ (the broken line represents the present result of equation (28))

on the evolution in the vicinity of the deterministic trajectory [11]. Comparing this with the paper by Mizerski [7] one can see that an inappropriate small-time description makes the evolution faster, causing the curves displaying the time dependence of the mean $\langle x\rangle$ and its variance $\langle(\Delta x)^2\rangle$ to be shifted to the left, even for a small noise parameter. Thus we can see the importance of a proper description of the early stage of the evolution.

It results from the above that the proposed method describes quite well the decay of an unstable state far enough from the critical region. A Gaussian approximation around the deterministic path used above is possible if the relation $q \ll 1$ holds. Since the stationary solution of (3) depends on x like a Gaussian function, our approximate procedure can give the exact stationary expressions. Here we can also notice the difference between the present method and that of Dekker [4], where an assumption about Gaussian fluctuations around the deterministic path is also employed. Since Dekker used $\tilde{x}$ as a variable he obtained an approximate formula for the mean $\langle x\rangle = \langle \tilde{x}^2\rangle$ only, and not for the variance $\langle(\Delta x)^2\rangle$.

The proposed formalism seems to be especially useful for calculating the multi-time correlation functions. For example, the two-time correlation function is given, by the definition, as

$$g(t_1, t_2) = \int_0^\infty \mathrm{d}x\, x \exp[L(x)(t_1 - t_2)] x W(x, t_2) \qquad t_1 \geq t_2. \tag{29}$$

One may check that the drift part $L_0(x)$ is the leading part of the FP operator $L(x)$ (4) in the evolution operator $\exp[L(x)(t_1 - t_2)]$. Thus we must extract the deterministic evolution operator $\exp[L_0(x)(t_1 - t_2)]$ to the right from it. The remaining operator treated like the operator $T(x, t)$ (13), gives in the lowest-order approximation a formula similar to (21) but with the function

$$\begin{aligned}\sigma_c(y, \tau) &= \int_0^\tau \mathrm{d}s\, 8\psi^2(y, -s)\phi(y, -s)\\ &= 8y[\tfrac{1}{2}(1-y)^3(\mathrm{e}^{2t}-1) + 3y(1-y)^2 t + \tfrac{3}{2}y^2(y-1)(1-\mathrm{e}^{-2t}) + \tfrac{1}{4}y^3(1-\mathrm{e}^{-4t})]\end{aligned} \tag{30}$$

instead of σ. Using this approximation we have

$$g(t_1, t_2) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \exp[\tfrac{1}{2}q\sigma_c(y, t_1 - t_2)] \frac{\partial^2}{\partial x^2} \delta(x - y) \exp[L_0(y)(t_1 - t_2)] y W(y, t)$$
$$= \int_0^\infty \mathrm{d}y\, \phi(y, t_1 - t_2) y W(y, t_2) \tag{31}$$

where the probability density function $W(y, t_2)$ is given by (25). One can notice that in the above formula (31) the fluctuations enter through $W(y, t_2)$ only. In order to incorporate the fluctuations into the terms depending on the time difference $t_1 - t_2$ one must use a higher-order approximation to calculate the evolution operator $\exp[L(x)(t_1 - t_2)]$.

The present method can easily be generalised for an arbitrary problem given by a FP equation. The only restriction is that the stationary solution may be approximated by a Gaussian function for an adequately chosen variable which means that the system does not evolve in the critical region of parameters.

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